



Spin squeezing in symmetric multiqubit states with two non-orthogonal Majorana spinors

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Received: 18 August 2018 / Accepted: 6 March 2019
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Abstract

Enhanced precision measurements using entangled many particle states are crucial for their technological applications in quantum information science and metrology. Squeezed spin states are a class of permutation symmetric N particle entangled states, which exhibit reduced quantum fluctuation in their collective spin angular momentum in a certain direction, and they are useful for quantum enhanced metrology. Permutation symmetric states attract attention as they offer significant test grounds for the description of entanglement in multipartite quantum systems, which is crucial for processing complex quantum information tasks. Spin squeezing serves as an experimentally amenable collective criterion of entanglement in symmetric multiqubit systems. In this paper, we explore spin-squeezing behavior in different classes of N -qubit symmetric states consisting of all permutations of two distinct spinors. We employ Majorana geometric representation of multiqubit states obeying exchange symmetry for this purpose. We prove that N qubit symmetric states consisting of two distinct non-orthogonal spinors do exhibit spin squeezing, thus expanding the avenues of their applicability in quantum enhanced sensing tasks.

Keywords Spin squeezing · Majorana geometric representation · Symmetric multiqubit states

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1 Introduction

Quantum states of N two-level atoms (qubits) obeying exchange symmetry were introduced by Dicke [1] in 1954 to explain the phenomenon of superradiance in quantum optics. Recent years have witnessed revolutionary experimental progress [2–6] in controlled generation of multiqubit Dicke states in different physical systems such as photons, cold atoms and trapped ions. Entanglement in multiqubit systems is a useful resource in quantum metrology and in quantum information science. Witnessing entanglement in multiqubit systems containing about 10^3 to 10^{12} atoms, that can only be addressed collectively, is a difficult task [7]. Collective measurements of the mean value and variance of total spin of the system aid experimental detection of entanglement in permutation symmetric ensemble of atoms. An established experiment-friendly criterion of entanglement in permutation symmetric N -qubit systems is *spin squeezing*, which manifests in the form of reduced variance of collective spin of the system, in a specific direction, below the value $N/4$ set by uncorrelated spin coherent state [8–34]. Correlations present in spin-squeezed states find potential applications in quantum enhanced metrology [35,36]. This has triggered many experiments generating spin-squeezed states in different platforms [37–47].

Formally, a permutation symmetric ensemble of N qubits is treated as a spin- $N/2$ system characterized by collective spin angular momentum components $\hat{J}_i = \sum_{k=1}^N \frac{\hat{\sigma}_i^{(k)}}{2}$, $i = x, y, z$ where $\hat{\sigma}_i^{(k)}$ are the Pauli matrices of the k^{th} qubit. In an uncorrelated state (spin coherent state), minimum value of variance $(\Delta J_{\perp})^2 = \langle J_{\perp}^2 \rangle - \langle J_{\perp} \rangle^2$ of collective spin in the direction orthogonal to the mean spin direction is $N/4$. Quantum correlations between elementary spin-1/2 particles could manifest via reduced value of variance $(\Delta J_{\perp})_{\min}^2$ below $N/4$.

Various definitions of spin squeezing have been proposed in the literature in different contexts [32]. We confine our attention to the one proposed originally by Kitagawa and Ueda [11], which captures the essence of spin squeezing in a most natural way. Spin-squeezing parameter offers a simple experimental criterion for the detection of entanglement in many particle systems, where it is not possible to address individual particles. Kitagawa and Ueda introduced a *spin-squeezing parameter* [11]

$$\xi = \frac{2(\Delta J_{\perp})_{\min}}{\sqrt{N}} \quad (1)$$

which is equal to 1 in a spin coherent state and is less than 1 in a spin-squeezed state. In symmetric multiqubit states, the criterion $\xi < 1$ implies entanglement, but the converse is not in general true [28,48,49].

Mathematical treatment of symmetric multiqubit system is simpler because a N qubit system, obeying exchange symmetry, gets confined to the $N + 1$ -dimensional subspace of the 2^N -dimensional Hilbert space. This symmetric subspace is spanned by the Dicke states, $\{|\frac{N}{2}, \frac{N}{2} - r\rangle, r = 0, 1, 2, \dots, N\}$ which are simultaneous eigenstates of collective spin operators \hat{J}^2, \hat{J}_z . Dicke states (except in the case of $r = 0, N$) consist of permutations of two distinct orthogonal *spinors* $|0\rangle$ (spin-up) and $|1\rangle$ (spin-down), which can be represented by diagonally opposite points on the Bloch sphere.

An elegant geometrical representation of multiqubit states was proposed by Majorana [50] in 1932. The Majorana geometric representation enables one to visualize multiqubit pure symmetric states composed of N distinct qubits (spinors) as constellation of N distinct points on the Bloch sphere [50]. While N -qubit Dicke states consist of two distinct orthogonal spinors $|0\rangle, |1\rangle$, a natural extension is to consider N -qubit symmetric states characterized by *two* non-orthogonal spinors. It is well known that Dicke states $\{|\frac{N}{2}, \frac{N}{2} - r\rangle, r = 0, 1, 2, \dots, N\}$ (except in the case of $r = 0$ and $r = N$) are entangled, but are not spin squeezed [28,49]. It has also been recognized that N -qubit symmetric states, consisting of permutations of two or more distinct spinors, are always entangled [51–54] but their spin-squeezing property is not explored so far. In this paper, we investigate spin-squeezing behavior of different classes of N -qubit symmetric states consisting of two distinct spinors—denoted by $\{\mathcal{D}_{N-k,k}\}$, $k = 1, 2, \dots, (N-1)/2$ (for odd N) or $N/2$ (for even N)—where one of the spinors occurs k times and the other $(N-k)$ times [51–53]. We show that spin-squeezing parameter ξ [see Eq. (1)] is less than 1 for those states of the family $\{\mathcal{D}_{N-k,k}\}$ with two distinct non-orthogonal spinors, indicating that they are spin squeezed. Spin-squeezing behavior of this class of states has not been explored so far. Thus, our investigation opens up applicability of this class of collective spin states in quantum enabled sensing.

This paper is organized as follows: Sect. 2 gives a brief overview of the Majorana geometric representation of symmetric N -qubit states and provides a canonical structure for the family of states $\{\mathcal{D}_{N-k,k}\}$, $k = 1, 2, \dots, (N-1)/2$ (for odd N) or $N/2$ (for even N) consisting of two distinct spinors. In Sect. 3, we first express the spin-squeezing parameter ξ for states belonging to the family $\{\mathcal{D}_{N-k,k}\}$ in terms of the elements of their two-qubit density matrix. We then evaluate the two-qubit density matrices of the N -qubit states belonging to $\{\mathcal{D}_{N-k,k}\}$, for each k . On evaluating the spin-squeezing parameter using the corresponding reduced two-qubit density matrices in Sect. 3, we show that the states belonging to the family $\{\mathcal{D}_{N-k,k}\}$ are spin squeezed when the two constituent spinors are not orthogonal. Section 4 gives a brief summary of results.

2 Majorana representation of pure symmetric multiqubit states

In the novel 1932 paper [50], Ettore Majorana proposed that a quantum system prepared in a pure spin $j = \frac{N}{2}$ state can be represented as a permutation of the states of N constituent spinors (qubits) as follows:

$$|\Psi_{\text{sym}}\rangle = \mathcal{N} \sum_P \hat{P} \{|\epsilon_1, \epsilon_2, \dots, \epsilon_N\rangle\}, \quad (2)$$

where

$$|\epsilon_r\rangle = a_r |0\rangle + b_r e^{i\beta_r} |1\rangle, \quad r = 1, 2, \dots, N, \quad a_r^2 + b_r^2 = 1 \quad (3)$$

denote the spinors constituting the symmetric N -qubit state $|\Psi_{\text{sym}}\rangle$; \hat{P} corresponds to the set of all $N!$ permutations and \mathcal{N} denotes normalization factor. Equation (2) is

referred to as the *Majorana geometric representation* of a pure quantum state $|\Psi_{\text{sym}}\rangle$ of spin $j = N/2$. Equivalently, $|\Psi_{\text{sym}}\rangle$ in Eq. (2) represents the permutationally symmetric N qubit state, expressed in terms of the constituent spinors $|\epsilon_r\rangle$, $r = 1, 2, \dots, N$. It may be seen that when all the N spinors $|\epsilon_r\rangle$, $r = 1, 2, \dots, N$ are identical, the state $|D_N\rangle = |\epsilon, \epsilon, \dots, \epsilon\rangle$ is separable. Symmetric N qubit states consisting of two distinct spinors $|\epsilon_1\rangle$, $|\epsilon_2\rangle$ are denoted by $\{D_{N-k,k}\}$ with representative states given by,

$$|D_{N-k,k}\rangle = \mathcal{N} \sum_P \hat{P} \left\{ \underbrace{|\epsilon_1, \epsilon_1, \dots, \epsilon_1\rangle}_{N-k}, \underbrace{|\epsilon_2, \epsilon_2, \dots, \epsilon_2\rangle}_k \right\} \quad (4)$$

where $k = 1, 2, \dots, (N-1)/2$ (for odd N) or $N/2$ (for even N). Dicke states $|\frac{N}{2}, \frac{N}{2} - r\rangle$, with $0 < r < N$ are representative states of the family $\{D_{N-k,k}\}$, containing permutations of two *orthogonal* spinors $|\epsilon_1\rangle = |0\rangle$, $|\epsilon_2\rangle = |1\rangle$. In general, a symmetric state $|D_{N-k,k}\rangle$ of N qubits belonging to the family $\{D_{N-k,k}\}$ of two distinct spinors can be reduced to a canonical form, characterized by only one real parameter [51,55], with the help of identical local unitary transformations on individual qubits. More specifically, a symmetric pure state $|D_{N-k,k}\rangle$, given in Eq. (4), is equivalent (under local unitary transformations) to the canonical state [51,55]:

$$|D_{N-k,k}\rangle \equiv \sum_{r=0}^k \beta_r^{(k)} \left| \frac{N}{2}, \frac{N}{2} - r \right\rangle, \quad (5)$$

$$\beta_r^{(k)} = \mathcal{N} \sqrt{\frac{N!(N-r)!}{r!}} \frac{a^{k-r} b^r}{(N-k)!(k-r)!}$$

consisting of permutations of two distinct spinors $|\epsilon'_1\rangle = |0\rangle$ and $|\epsilon'_2\rangle = a|0\rangle + b|1\rangle$ where $0 \leq a < 1$ is a real positive parameter and $b = \sqrt{1-a^2}$. Dicke states, characterized by two *orthogonal* spinors $|0\rangle$, $|1\rangle$, have $a = 0$. When $a = 1$, the N -qubit symmetric state is characterized by a single spinor $|0\rangle$ and corresponds to a separable state. The states $|D_{N-k,k}\rangle$ with $0 < a < 1$ are the ones characterized by *non-orthogonal* spinors.

While symmetric N -qubit states consisting of permutations of two or more distinct spinors are shown to be entangled [51–53], it is not known whether this entanglement manifests itself in terms of collective spin-squeezing behavior. In this context, it has been shown that Dicke states (constructed by permutation of two distinct orthogonal spinors) are entangled—but they are not spin squeezed [28,49]. Thus, it is of importance to investigate spin-squeezing property of the states $|D_{N-k,k}\rangle$ characterized by a general class of N qubit permutation symmetric states consisting of two distinct, non-orthogonal spinors.

3 Spin squeezing in N -qubit pure states characterized by two distinct spinors

Spin squeezing being a collective property of an ensemble of symmetric qubits, one needs to find mean value and variances of collective angular momentum operator to

evaluate the spin-squeezing parameter ξ [see Eq. (1)]. It has been shown that the mean value, variances of total angular momentum, and hence the spin-squeezing parameter ξ of an N qubit symmetric state can be expressed entirely in terms of the correlation matrix elements of its reduced two-qubit density matrix [24,28]. By construction, this method of evaluating spin-squeezing parameter is *exactly identical* to the usual way of calculating them in terms of the average and variances of the collective total angular momentum explicitly.

In Ref. [28], it is shown that the spin-squeezing parameter ξ [see Eq. (1)] of a N -qubit symmetric state can be expressed as,

$$\xi = \left[1 + (N - 1)(\tilde{n}_\perp T \hat{n}_\perp)_{\min} \right]^{1/2}. \quad (6)$$

where $T = (t_{ij})$, $i, j = x, y, z$ denotes the real symmetric 3×3 correlation matrix corresponding to the reduced two-qubit density matrix of a symmetric N -qubit state. If ρ denotes the reduced two-qubit density matrix of a symmetric N -qubit state, obtained by tracing over any of its $N - 2$ qubits, the correlation matrix elements t_{ij} are given by

$$t_{ij} = \text{Tr} [\rho (\sigma_i \otimes \sigma_j)] = \text{Tr} [\rho (\sigma_j \otimes \sigma_i)] \quad (7)$$

where σ_i , $i = x, y, z$ are the Pauli spin matrices. The unit vector \hat{n}_\perp in Eq.(6) is perpendicular to the mean spin vector \hat{n}_0 given by

$$\hat{n}_0 = \frac{\langle \hat{J} \rangle}{|\langle \hat{J} \rangle|} = \frac{(\langle \hat{J}_x \rangle, \langle \hat{J}_y \rangle, \langle \hat{J}_z \rangle)}{\sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_y \rangle^2 + \langle \hat{J}_z \rangle^2}} \quad (8)$$

and \tilde{n}_\perp is the transpose of \hat{n}_\perp . As the mean spin vector of all the qubits in a symmetric multiqubit state are identical, one has

$$\hat{n}_0 = \frac{(s_x, s_y, s_z)}{\sqrt{s_x^2 + s_y^2 + s_z^2}} \quad (9)$$

with

$$s_i = \text{Tr} [\rho (\sigma_i \otimes I)] = \text{Tr} [\rho (I \otimes \sigma_i)], \quad i = x, y, z \quad (10)$$

being the components of the qubit orientation \mathbf{s} in the two-qubit subsystem ρ .

Choosing a suitable co-ordinate system with mutually orthogonal triads \hat{n}_1 , \hat{n}_2 , \hat{n}_0 of basis vectors, such that the Z-axis is aligned along the mean spin direction \hat{n}_0 , it may be seen that the quadratic form $(\tilde{n}_\perp T \hat{n}_\perp)_{\min}$ is the *minimum eigenvalue* [24,28,34] of the 2×2 block T_\perp of the correlation matrix T in the basis \hat{n}_1, \hat{n}_2 orthogonal to the mean spin vector \hat{n}_0 .

$$\begin{aligned} (\tilde{n}_\perp T \hat{n}_\perp)_{\min} = \frac{1}{2} \left[(\tilde{n}_1 T \hat{n}_1 + \tilde{n}_2 T \hat{n}_2) \right. \\ \left. - \sqrt{(\tilde{n}_1 T \hat{n}_1 - \tilde{n}_2 T \hat{n}_2)^2 + 4 (\tilde{n}_1 T \hat{n}_2)^2} \right]. \quad (11) \end{aligned}$$

Consequently, the spin-squeezing parameter ξ can be expressed in an operationally simple form, on substituting Eq. (11) into Eq. (6). In other words, the Kitegawa–Ueda spin-squeezing parameter ξ may be evaluated using the reduced two-qubit density matrix of any random pair of qubits of a N -qubit symmetric system.

In the following subsections, we evaluate the general form of the reduced two-qubit density matrix of any random pair of qubits of the state $|D_{N-k,k}\rangle$ [see Eq. (5)], belonging to the family $\{D_{N-k,k}\}$ and deduce their spin-squeezing property.

3.1 Evaluation of two-qubit density matrices of the state $|D_{N-k,k}\rangle$

The two-qubit density matrix $\rho^{(k)}$ corresponding to any random pair of qubits in the state $|D_{N-k,k}\rangle \in \{D_{N-k,k}\}$ is given by,

$$\begin{aligned} \rho^{(k)} &= \text{Tr}_{N-2} (|D_{N-k,k}\rangle\langle D_{N-k,k}|) \\ &= \text{Tr}_{N-2} \left\{ \sum_{r,r'=0}^k \beta_r^{(k)} \beta_{r'}^{(k)} \sum_{m_2,m'_2} \left[c_{m_2}^{(r)} c_{m'_2}^{(r')} \left| \frac{N}{2} - 1, \frac{N}{2} - r - m_2 \right\rangle \right. \right. \\ &\quad \left. \left. \left\langle \frac{N}{2} - 1, \frac{N}{2} - r' - m'_2 \right| \otimes |1, m_2\rangle\langle 1, m'_2| \right] \right\} \\ &= \sum_{m_2,m'_2=1,0,-1} \rho_{m_2,m'_2}^{(k)} |1, m_2\rangle\langle 1, m'_2|, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \rho_{m_2,m'_2}^{(k)} &= \sum_{r,r'=0}^k \beta_r^{(k)} \beta_{r'}^{(k)} c_{m_2}^{(r)} c_{m'_2}^{(r')} \left\{ \sum_{m_1=(-N/2)+1}^{(N/2)-1} \left\langle \frac{N}{2} - 1, m_1 \right| \frac{N}{2} - 1, \frac{N}{2} - r - m_2 \right\rangle \right. \\ &\quad \left. \left\langle \frac{N}{2} - 1, \frac{N}{2} - r' - m'_2 \right| \frac{N}{2} - 1, m_1 \right\rangle \right\} \end{aligned} \tag{13}$$

The associated Clebsch–Gordan coefficients $c_{m_2}^{(r)} = C\left(\frac{N}{2} - 1, 1, \frac{N}{2}; m - m_2, m_2, m\right)$, $m = \frac{N}{2} - r, m_2 = 1, 0, -1$ are given explicitly by [56]

$$\begin{aligned} c_1^{(r)} &= \sqrt{\frac{(N-r)(N-r-1)}{N(N-1)}}, \quad c_{-1}^{(r)} = \sqrt{\frac{r(r-1)}{N(N-1)}}, \\ c_0^{(r)} &= \sqrt{\frac{2r(N-r)}{N(N-1)}} \end{aligned} \tag{14}$$

By expressing $\rho^{(k)}$ in the standard two-qubit basis $\{|0_A, 0_B\rangle, |0_A, 1_B\rangle, |1_A, 0_B\rangle, |1_A, 1_B\rangle\}$, (using the relations between angular momentum basis $|1, m_2 = \pm 1, 0\rangle$ and the

local qubit basis, i.e., $|1, 1\rangle = |0_A, 0_B\rangle$, $|1, 0\rangle = (|0_A, 1_B\rangle + |1_A, 0_B\rangle)/\sqrt{2}$, $|1, -1\rangle = |1_A, 1_B\rangle$, one obtains the following simplified form [26,57] for the symmetric two-qubit reduced density matrix:

$$\rho^{(k)} = \begin{pmatrix} A^{(k)} & B^{(k)} & B^{(k)} & C^{(k)} \\ B^{(k)} & D^{(k)} & D^{(k)} & E^{(k)} \\ B^{(k)} & D^{(k)} & D^{(k)} & E^{(k)} \\ C^{(k)} & E^{(k)} & E^{(k)} & F^{(k)} \end{pmatrix}, \quad (15)$$

where $A^{(k)}$, $B^{(k)}$, $C^{(k)}$, $D^{(k)}$, $E^{(k)}$ and $F^{(k)}$ are real.

Now, we proceed to discuss spin squeezing in detail in the illustrative cases $k = 1, 2$ in the family of states $\{\mathcal{D}_{N-k, k}\}$.

3.2 Spin squeezing in the class of states $\{\mathcal{D}_{N-1, 1}\}$

The reduced two-qubit density matrix $\rho^{(1)}$ drawn from the N -qubit pure states of the family $\{\mathcal{D}_{N-1, 1}\}$ [see Eq. (12)] has the following explicit structure:

$$\begin{aligned} \rho^{(1)} &= \text{Tr}_{N-2} (|D_{N-1, 1}\rangle\langle D_{N-1, 1}|) \\ &= \left((\beta_0^{(1)})^2 + (\beta_1^{(1)} c_1^{(1)})^2 \right) |1, 1\rangle\langle 1, 1| \\ &\quad + (\beta_1^{(1)} c_0^{(1)})^2 |1, 0\rangle\langle 1, 0| + \beta_0^{(1)} \beta_1^{(1)} c_0^{(1)} |1, 1\rangle\langle 1, 0| \\ &\quad + \beta_0^{(1)} \beta_1^{(1)} c_0^{(1)} |1, 0\rangle\langle 1, 1| \end{aligned} \quad (16)$$

Here, [see Eq. (5)] we have $\beta_0^{(1)} = \mathcal{N} N a$, $\beta_1^{(1)} = \mathcal{N} \sqrt{N(1-a^2)}$ with $\mathcal{N} = \frac{1}{\sqrt{N^2 a^2 + N(1-a^2)}}$ and the associated nonzero Clebsch–Gordan coefficients [see Eq. (14)] are given by

$$c_1^{(1)} = \sqrt{\frac{N-2}{N}}, \quad c_0^{(1)} = \sqrt{\frac{2}{N}}. \quad (17)$$

Furthermore, in the standard two-qubit basis $\{|0_A, 0_B\rangle, |0_A, 1_B\rangle, |1_A, 0_B\rangle, |1_A, 1_B\rangle\}$, we obtain

$$\rho^{(1)} = \begin{pmatrix} A^{(1)} & B^{(1)} & B^{(1)} & 0 \\ B^{(1)} & D^{(1)} & D^{(1)} & 0 \\ B^{(1)} & D^{(1)} & D^{(1)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$A^{(1)} = \frac{N^2 a^2 + (N-2)(1-a^2)}{N^2 a^2 + N(1-a^2)}, \quad B^{(1)} = \frac{a\sqrt{1-a^2}}{1+a^2(N-1)},$$

$$D^{(1)} = \frac{1 - a^2}{N^2 a^2 + N(1 - a^2)}, \tag{18}$$

We obtain the qubit orientations [see Eq. (10)]

$$s_x = 2 B^{(1)}, \quad s_y = 0, \quad s_z = A^{(1)}$$

using which we find an orthogonal triad of basis vectors

$$\begin{aligned} \hat{n}_0 &= \left(\frac{s_x}{\sqrt{s_x^2 + s_z^2}}, 0, \frac{s_z}{\sqrt{s_x^2 + s_z^2}} \right), \\ \hat{n}_1 &= (0, 1, 0), \\ \hat{n}_2 &= \left(-\frac{s_z}{\sqrt{s_x^2 + s_z^2}}, 0, \frac{s_x}{\sqrt{s_x^2 + s_z^2}} \right) \end{aligned}$$

with \hat{n}_0 denoting the mean spin direction. On simplifying, we obtain [see Eq. (11)] $\tilde{n}_1 T \hat{n}_2 = \tilde{n}_2 T \hat{n}_1 = 0$ and $(\tilde{n}_\perp T \hat{n}_\perp)_{\min} = \tilde{n}_2 T \hat{n}_2$. Thus, the corresponding spin-squeezing parameter takes the form¹ [see (Eq. 6)]

$$\xi = \sqrt{1 + (N - 1) (\tilde{n}_2 T \hat{n}_2)} \tag{19}$$

where

$$\tilde{n}_2 T \hat{n}_2 = \frac{2 \left[(A^{(1)})^2 D^{(1)} - 2 (B^{(1)})^2 \right]}{4 (B^{(1)})^2 + (A^{(1)})^2}. \tag{20}$$

In Fig. 1, we have plotted ξ , for the states in the family $\{\mathcal{D}_{N-1,1}\}$, as a function of the parameter a and number of qubits N . It can be seen that the states of the family $|D_{N-1,1}\rangle$ are spin squeezed, except when $a = 0$.

3.3 Spin squeezing in the class $\{\mathcal{D}_{N-2,2}\}$

When $k = 2$, the state $|D_{N-k,k}\rangle$ has the form [see Eq. (5)]

$$|D_{N-2,2}\rangle = \beta_0^{(2)} \left| \frac{N}{2}, \frac{N}{2} \right\rangle + \beta_1^{(2)} \left| \frac{N}{2}, \frac{N}{2} - 1 \right\rangle + \beta_2^{(2)} \left| \frac{N}{2}, \frac{N}{2} - 2 \right\rangle \tag{21}$$

¹ For states belonging to the class $\{\mathcal{D}_{N-k,k}\}$ with different values of $k = 2, 3, \dots$, we find that (i) the mean spin direction (denoted by the unit vector \hat{n}_0) lies in the XZ plane and (ii) the 2×2 block T_\perp of correlation matrix T , which is expressed in the basis (\hat{n}_1, \hat{n}_2) orthogonal to the mean spin direction \hat{n}_0 , is diagonal, i.e., $\tilde{n}_1 T \hat{n}_2 = 0$. It is seen that $\min(\tilde{n}_1 T \hat{n}_1, \tilde{n}_2 T \hat{n}_2) = \tilde{n}_2 T \hat{n}_2$. Thus, the spin-squeezing parameter [see Eqs. (6) and (11)] takes the form $\xi = \sqrt{1 + (N - 1) (\tilde{n}_2 T \hat{n}_2)}$ for any k and N .

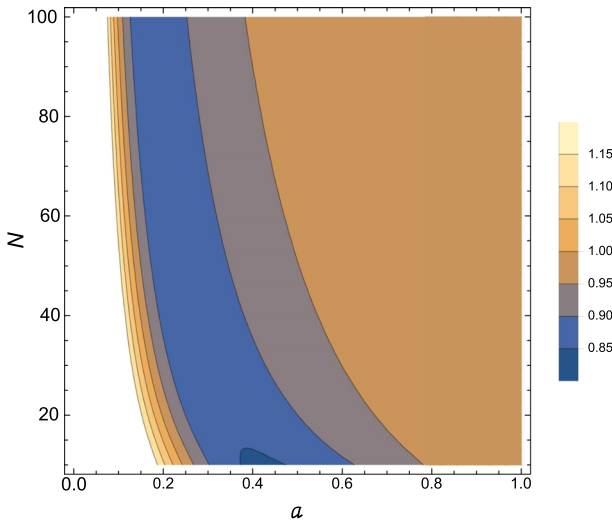


Fig. 1 Spin-squeezing parameter ξ as a function of the parameter a and number of qubits N in the class $\{\mathcal{D}_{N-1,1}\}$ of N -qubit pure symmetric states with two distinct spinors

where

$$\begin{aligned} \beta_0^{(2)} &= \mathcal{N} \frac{N(N-1)}{2} a^2, & \beta_1^{(2)} &= \mathcal{N} \sqrt{N(N-1)} a \sqrt{1-a^2}, \\ \beta_2^{(2)} &= \mathcal{N} \sqrt{\frac{N(N-1)}{2}} (1-a^2). \end{aligned} \tag{22}$$

and \mathcal{N} , the normalization factor, satisfies the relation $(\beta_0^{(2)})^2 + (\beta_1^{(2)})^2 + (\beta_2^{(2)})^2 = 1$. Following the procedure outlined in Sect. 3.1, we evaluate the two-qubit reduced density matrix $\rho^{(2)} = \text{Tr}_{N-2} (|D_{N-2,2}\rangle\langle D_{N-2,2}|)$ and express it in the standard two-qubit basis $\{|0_A, 0_B\rangle, |0_A, 1_B\rangle, |1_A, 0_B\rangle, |1_A, 1_B\rangle\}$:

$$\rho^{(2)} = \begin{pmatrix} A^{(2)} & B^{(2)} & B^{(2)} & C^{(2)} \\ B^{(2)} & D^{(2)} & D^{(2)} & E^{(2)} \\ B^{(2)} & D^{(2)} & D^{(2)} & E^{(2)} \\ C^{(2)} & E^{(2)} & E^{(2)} & F^{(2)} \end{pmatrix}.$$

Here, we have

$$\begin{aligned} A^{(2)} &= (\beta_0^{(2)})^2 + (\beta_1^{(2)} c_1^{(1)})^2 + (\beta_2^{(2)} c_1^{(2)})^2, \\ B^{(2)} &= \frac{\beta_0^{(2)} \beta_1^{(2)} c_0^{(1)} + \beta_1^{(2)} \beta_2^{(2)} c_1^{(1)} c_0^{(2)}}{\sqrt{2}}, \\ C^{(2)} &= \beta_0^{(2)} \beta_2^{(2)} c_{-1}^{(2)}, \end{aligned}$$

$$\begin{aligned}
 D^{(2)} &= \frac{\left(\beta_1^{(2)} c_0^{(1)}\right)^2 + \left(\beta_2^{(2)} c_0^{(2)}\right)^2}{2}, \\
 E^{(2)} &= \frac{\beta_1^{(2)} \beta_2^{(2)} c_0^{(1)} c_{-1}^{(2)}}{\sqrt{2}}, \\
 F^{(2)} &= \left(\beta_2^{(2)} c_{-1}^{(2)}\right)^2.
 \end{aligned}$$

where some of the nonzero Clebsch–Gordan coefficients [see Eq. (14)] are given in Eq. (17) and the remaining are given by

$$\begin{aligned}
 c_1^{(2)} &= \sqrt{\frac{(N-3)(N-2)}{N(N-1)}}, & c_0^{(2)} &= 2\sqrt{\frac{N-2}{N(N-1)}} \\
 c_{-1}^{(2)} &= \sqrt{\frac{2}{N(N-1)}}.
 \end{aligned} \tag{23}$$

Substituting for $\beta_i^{(2)}$, $i = 0, 1, 2$ and the Clebsch–Gordan coefficients, we obtain the density matrix $\rho^{(2)}$ in terms of the number N of qubits, and the real parameter a . We then evaluate the spin-squeezing parameter ξ following the same procedure followed in Sect. 3.2, while discussing the class $\{\mathcal{D}_{N-1,1}\}$. We identify that the mean spin direction \hat{n}_0 lies in the XZ plane and the element of correlation matrix $\tilde{n}_1 T \hat{n}_2 = 0$. This facilitates the evaluation of the spin-squeezing parameter to be $\xi = \sqrt{1 + (N-1) (\tilde{n}_2 T \hat{n}_2)}$. We have plotted ξ as a function of the parameter a for different values of N in Fig 2.

In general, for any arbitrary k , we evaluate the two-qubit density matrix $\rho^{(k)} = \text{Tr}_{N-2} (|D_{N-k,k}\rangle\langle D_{N-k,k}|)$, in the standard two-qubit basis, in the form given in Eq. (15) with elements $A^{(k)}, B^{(k)}, C^{(k)}, D^{(k)}, E^{(k)}, F^{(k)}$ given by,

$$\begin{aligned}
 A^{(k)} &= \sum_{r=0}^k \left(\beta_r^{(k)} c_1^{(r)}\right)^2, \\
 B^{(k)} &= \frac{1}{\sqrt{2}} \sum_{r=0}^{k-1} \beta_r^{(k)} \beta_{r+1}^{(k)} c_1^{(r)} c_0^{(r+1)}, \\
 C^{(k)} &= \sum_{r=0}^{k-2} \beta_r^{(k)} \beta_{r+2}^{(k)} c_1^{(r)} c_{-1}^{(r+2)}, \\
 D^{(k)} &= \frac{1}{2} \sum_{r=0}^{k-1} \left(\beta_{r+1}^{(k)}\right)^2 c_0^{(r+1)2}, \\
 E^{(k)} &= \frac{1}{\sqrt{2}} \sum_{r=0}^{k-2} \beta_{r+1}^{(k)} \beta_{r+2}^{(k)} c_0^{(r+1)} c_{-1}^{(r+2)},
 \end{aligned}$$

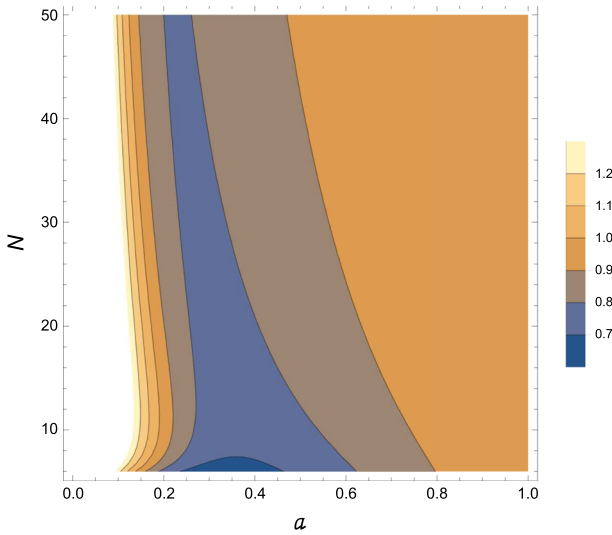


Fig. 2 Spin-squeezing parameter ξ as a function of the parameter a and number of qubits N in the class $\{\mathcal{D}_{N-2,2}\}$ of pure symmetric states

$$F^{(k)} = \sum_{r=0}^{k-2} \left(\beta_{r+2}^{(k)} c_{-1}^{(r+2)} \right)^2 .$$

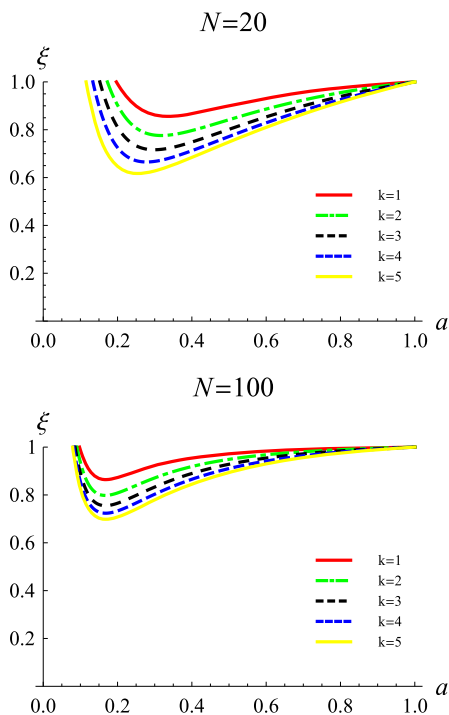
The mean spin direction \hat{n}_0 of the qubits in $|D_{N-k,k}\rangle$ lies in the XZ plane, and the spin-squeezing parameter ξ for any arbitrary state $|D_{N-k,k}\rangle$ belonging to the family $\{\mathcal{D}_{N-k,k}\}$ can be readily evaluated using $\xi = \sqrt{1 + (N - 1) (\tilde{n}_2 T \hat{n}_2)}$ where

$$\begin{aligned} \tilde{n}_2 T \hat{n}_2 = & \frac{2 (A^{(k)} - F^{(k)})^2 (C^{(k)} + D^{(k)})}{\left[4 (B^{(k)} + E^{(k)})^2 + (A^{(k)} - F^{(k)})^2 \right]} \\ & + \frac{4 (B^{(k)} + E^{(k)})^2 (1 - 4D^{(k)})}{\left[4 (B^{(k)} + E^{(k)})^2 + (A^{(k)} - F^{(k)})^2 \right]} \\ & - \frac{8 (A^{(k)} - F^{(k)}) \left((B^{(k)})^2 - (E^{(k)})^2 \right)}{\left[4 (B^{(k)} + E^{(k)})^2 + (A^{(k)} - F^{(k)})^2 \right]} . \end{aligned}$$

In Fig. 3, we have illustrated the variation of the spin-squeezing parameter ξ in the N -qubit pure symmetric state $|D_{N-k,k}\rangle$ with different k and N .

It can be readily seen from Fig. 3 that for a fixed N , the collective spin-squeezing parameter of the state $|D_{N-k,k}\rangle$ reduces with the increase in k . In other words, the N qubit states $|D_{N-k,k}\rangle$ exhibit higher spin squeezing for larger values of k . This is consistent with the fact that entanglement between constituent qubits sharing the

Fig. 3 Spin-squeezing parameter ξ of states $|D_{N-k,k}\rangle$ [see (Eq. 5)] for different values of $k = 1, 2, 3, 4, 5$ and for number of qubits $N = 20$ and $N = 100$



state $|D_{N-k,k}\rangle$ increases with the increase in k , attaining a maximum, respectively, for $k = N/2$ for even N , and $k = (N - 1)/2$ for odd N [51,54]. It is clearly seen that (i) the states $\{|D_{N-k,k}\rangle, 0 < a < 1\}$, characterized by two distinct non-orthogonal spinors, exhibit spin squeezing; (ii) for a given N , enhanced spin-squeezing behavior (equivalently, decrease in the parameter ξ below 1) is witnessed as k increases. This may be attributed to larger entanglement as k increases.

Recall that the one parameter family of N qubit symmetric states $\{|D_{N-k,k}\rangle\}$, reduce to the entangled (but non-squeezed) Dicke state $|N/2, N/2 - k\rangle, k = 1, 2, \dots$ and to the product state $|N/2, N/2\rangle = |0\rangle^{\otimes N}$ of qubits for the extreme values of the real parameter $a = 0, 1$. The spin-squeezing parameter ξ varies smoothly between the two end points $a = 0, 1$ (at these end points the parameter $\xi \geq 1$, i.e., there is no spin squeezing at $a = 0, 1$) and exhibits a valley as a function of the parameter $0 < a < 1$. The position of the valley shifts for different values of k , in agreement with the fact that, larger the spin squeezing (decrease in the value of ξ), higher the value of k —which is analogous to the behavior of entanglement [51,54]. .

4 Summary

We have explored spin squeezing in symmetric multiqubit pure states $|D_{N-k,k}\rangle$ characterized by two distinct *non-orthogonal* Majorana spinors. To evaluate the spin-squeezing parameter ξ of the state $|D_{N-k,k}\rangle$, we make use of the expression [see

Eq. (6)] for ξ given entirely in terms of the elements of the density matrix corresponding to any random pair of qubits of the state $|D_{N-k,k}\rangle$ [24]. We have used the canonical form of the state $|D_{N-k,k}\rangle$ which is characterized by a single real parameter a [see Eq. (5)] and divided the N -qubit state into two parts containing $N - 2$ and 2 qubits, respectively. By tracing over the $N - 2$ qubits, we obtain the density matrix corresponding to any two qubits of the state $|D_{N-k,k}\rangle$. The correlation matrix elements of the two-qubit density matrix, expressed in the basis perpendicular to the mean spin direction, lead to the evaluation the spin-squeezing parameter ξ . The variation of spin squeezing with respect to the real parameter $0 < a \leq 1$ characterizing the state $|D_{N-k,k}\rangle$ is graphically illustrated for different values of $k = 1, 2, \dots, (N - 1)/2$ (for odd N) or $N/2$ (for even N). While the Dicke states $|\frac{N}{2}, \frac{N}{2} - l\rangle$ constituted by permutations of two orthogonal spinors are not spin squeezed, our work brings out the interesting fact that their generalizations, viz., N -qubit symmetric states consisting of two *non-orthogonal* spinors, exhibit spin squeezing. As a result of recent technological innovations in producing spin-squeezed states and their potential applicability in diverse platforms such as high-precision spectroscopy, quantum enhanced metrology, we believe that our work opens up expanding avenues of experimental investigations on employing generalized Dicke-class of permutation symmetric states, in these areas of promise.

Acknowledgements KSA would like to thank the University Grants Commission for providing a BSR-RFSMS fellowship during the present work. ARU acknowledges the support of UGC MRP (Ref. MRP-MAJOR-PHYS-2013-29318). We thank Professor A. K. Rajagopal for insightful discussions.

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